

# PROPERTY $T$ OF REDUCED $C^*$ -CROSSED PRODUCTS BY DISCRETE GROUPS

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**ABSTRACT.** We generalize the main result of [6] and show that if  $G$  is an amenable discrete group with an action  $\alpha$  on a finite nuclear unital  $C^*$ -algebra  $A$  such that the reduced crossed product  $A \rtimes_{\alpha,r} G$  has property  $T$ , then  $G$  is finite and  $A$  is finite dimensional. As an application, an infinite discrete group  $H$  is non-amenable if and only if the uniform Roe algebra  $C_u^*(H)$  has property  $T$ .

## 1. INTRODUCTION

Property  $T$  for unital  $C^*$ -algebras was introduced by Bekka in [1] and was studied by different people (see e.g. [2, 6, 8, 9]). In particular, it was shown by Kamalov in [6] that

if  $G$  is a discrete amenable group acting on a commutative unital  $C^*$ -algebra  $A$  such that the crossed product has property  $T$ , then  $G$  is finite and  $A$  is finite dimensional.

The aims of this paper is to extend this result to the case of finite nuclear unital  $C^*$ -algebras, and to give an application of this result. As expected, a result of Brown in [2] is one of our main tools.

## 2. THE MAIN RESULTS

Throughout this article,  $G$  is a discrete group acting on a unital  $C^*$ -algebra  $A$  through an action  $\alpha$  (by automorphisms).

Let  $T(A)$  be the set of all traical states on  $A$ . For any  $\tau \in T(A)$ , we denote by  $\pi_\tau : A \rightarrow \mathcal{B}(\mathcal{H}_\tau)$  the GNS representation corresponding to  $\tau$  and by  $\xi_\tau$  a norm one cyclic vector in  $\mathcal{H}_\tau$  with

$$\tau(a) = \langle \pi_\tau(a)\xi_\tau, \xi_\tau \rangle \quad (a \in A).$$

Recall that  $A$  is said to be *finite* if  $T(A)$  separates points of  $A_+$  ([4, Theorem 3.4]). We also recall from [1, Remark 2] that if  $T(A) = \emptyset$ , then  $A$  has property  $T$ .

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We use  $T_\alpha(A)$  to denote the set of all  $\alpha$ -invariant traical states on  $A$ , and recall that  $A$  is said to be  $\alpha$ -finite if  $T_\alpha(A)$  separates points of  $A_+$  (see [4, Theorem 8.1]). We also denote by  $A \rtimes_{\alpha,r} G$  the reduced crossed product of  $\alpha$ , and identify  $A \subseteq A \rtimes_{\alpha,r} G$  as well as  $G \subseteq A \rtimes_{\alpha,r} G$  through their canonical embeddings

Let us first give the following well-known facts. Since we cannot find precise references for them, we present their simple arguments here.

**Lemma 1.** (a)  $T(A \rtimes_{\alpha,r} G) \neq \emptyset$  if and only if  $T_\alpha(A) \neq \emptyset$ .

(b) If  $A$  is  $\alpha$ -finite, then  $A \rtimes_{\alpha,r} G$  is finite.

(c) If  $G$  is amenable and  $T(A) \neq \emptyset$ , then  $T_\alpha(A) \neq \emptyset$ .

**Proof:** Let us denote  $B := A \rtimes_{\alpha,r} G$ , and consider  $\mathcal{E} : B \rightarrow A$  to be the canonical conditional expectation (see e.g. [3, Proposition 4.1.9]).

(a) If  $\sigma \in T(B)$ , then  $\sigma(\alpha_t(a)) = \sigma(tat^{-1}) = \sigma(a)$  ( $a \in A; t \in G$ ), which means that  $\sigma|_A \in T_\alpha(A)$ . Conversely, for any  $\tau \in T_\alpha(A)$  and any  $x = \sum_{s \in G} a_s s$  with  $a_s = 0$  except for a finite number of  $s$ , one has

$$\tau(\mathcal{E}(x^*x)) = \tau\left(\sum_{r \in G} \alpha_{r^{-1}}(a_r^* a_r)\right) = \tau\left(\sum_{r \in G} a_r a_r^*\right) = \tau(\mathcal{E}(xx^*)).$$

Hence,  $\tau \circ \mathcal{E}$  belongs to  $T(B)$ , because it is continuous.

(b) As  $\mathcal{E}$  is faithful, we know that  $B$  is a Hilbert  $A$ -module under the  $A$ -valued inner product

$$\langle x, y \rangle_A := \mathcal{E}(x^*y) \quad (x, y \in B).$$

Moreover, for any  $\tau \in T_\alpha(A)$ , if  $\pi_\tau^B$  is the canonical representation of  $B$  on the Hilbert space  $B \otimes_{\pi_\tau} \mathcal{H}_\tau$  (see e.g. [7, Proposition 4.5] for its definition; note that we identify a Hilbert  $\mathbb{C}$ -module with a Hilbert space by considering the conjugation of the inner product), then  $(B \otimes_{\pi_\tau} \mathcal{H}_\tau, \pi_\tau^B)$  coincides with  $(\mathcal{H}_{\tau \circ \mathcal{E}}, \pi_{\tau \circ \mathcal{E}})$  (observe that  $1 \otimes \xi_\tau$  is a cyclic vector for  $\pi_\tau^B$  with the state defined by  $1 \otimes \xi_\tau$  being  $\tau \circ \mathcal{E}$ ).

Let  $(\mathcal{H}_0, \pi_0) := \bigoplus_{\tau \in T_\alpha(A)} (\mathcal{H}_\tau, \pi_\tau)$ . Since  $A$  is  $\alpha$ -finite, one knows that  $\pi_0$  is faithful. It is easy to verify that the representation  $\pi_0^B$  of  $B$  on  $B \otimes_{\pi_0} \mathcal{H}_0$  induced by  $\pi_0$  is also faithful, and that  $\pi_0^B$  coincides with  $\bigoplus_{\tau \in T_\alpha(A)} \pi_\tau^B$ . Consequently,  $\bigoplus_{\tau \in T_\alpha(A)} (\mathcal{H}_{\tau \circ \mathcal{E}}, \pi_{\tau \circ \mathcal{E}})$  is faithful, which means that  $\{\tau \circ \mathcal{E} : \tau \in T_\alpha(A)\}$  (which is a subset of  $T(B)$  by the argument of part (a)) separates points of  $B_+$ .

(c) Note that  $T(A)$  is a non-empty weak\*-compact convex subset of  $A^*$  and  $\alpha$  induces an action of  $G$  on  $T(A)$  by continuous affine maps. Day's fixed point theorem (see [5, Theorem 1]) produces a fixed point  $\tau_0 \in T(A)$  for this action. Obviously,  $\tau_0 \in T_\alpha(A)$ .  $\square$

We warn the readers that part (c) of the above is not true for non-unital  $C^*$ -algebras.

Our main theorem concerns with the situation when  $A \rtimes_{\alpha,r} G$  is nuclear and has property  $T$ . In this situation, [2, Theorem 5.1] tells us that  $A \rtimes_{\alpha,r} G$  is a direct sum of a finite dimensional  $C^*$ -algebra and a nuclear  $C^*$ -algebra with no tracial state (note that although all  $C^*$ -algebras in [2] are assumed to be separable, [2, Theorem 5.1] is true in the non-separable case because one can use [3, Theorem 6.2.7] to replace [2, Theorem 4.2]). The following theorem implies that if  $G$  is infinite, then we arrive at one of the extreme that the whole reduced crossed product has no tracial state. This proposition, together with its proof, is a main ingredient in the argument for our main theorem.

**Proposition 2.** *Let  $G$  be an infinite discrete group acting on a unital  $C^*$ -algebra  $A$  through an action  $\alpha$ . If  $A \rtimes_{\alpha,r} G$  is nuclear and has property  $T$ , then  $T(A \rtimes_{\alpha,r} G) = \emptyset$ .*

**Proof:** Let  $I_\alpha := \bigcap_{\tau \in T_\alpha(A)} \ker \pi_\tau$  and  $A_\alpha := A/I_\alpha$ . Suppose on contrary that  $T(A \rtimes_{\alpha,r} G) \neq \emptyset$ . Then  $I_\alpha \neq A$  because of Lemma 1(a). As  $\ker \pi_\tau = \{x \in A : \tau(x^*x) = 0\}$  ( $\tau \in T(A)$ ), we know that  $I_\alpha$  is  $\alpha$ -invariant, and hence  $\alpha$  produces an action  $\beta$  of  $G$  on  $A_\alpha$ . Moreover, every element in  $T_\alpha(A)$  induces an element in  $T_\beta(A_\alpha)$ , which gives the  $\beta$ -finiteness of  $A_\alpha$ .

Since  $A_\alpha \rtimes_{\beta,r} G$  is a quotient  $C^*$ -algebra of  $A \rtimes_{\alpha,r} G$ , the hypothesis implies  $A_\alpha \rtimes_{\beta,r} G$  to be nuclear and having property  $T$ . Therefore, [2, Theorem 5.1] tells us that  $A_\alpha \rtimes_{\beta,r} G = C \oplus D$ , where  $C$  is finite dimensional and  $T(D) = \emptyset$ . However, the finiteness of  $A_\alpha \rtimes_{\beta,r} G$  (which follows from Lemma 1(b)) tells us that  $D = (0)$ . Consequently,  $A_\alpha \rtimes_{\beta,r} G$  is a non-zero finite dimensional  $C^*$ -algebra, which contradicts the fact that  $G$  is infinite.  $\square$

The following is our main theorem which concerns with the other extreme. More precisely, what we obtained is a situation (which include the one in [6]) under which the reduced crossed product is finite dimensional.

Notice that the finiteness assumption of  $A$  is indispensable. In fact, if  $A$  is the direct sum of  $\mathbb{C}$  with a nuclear unital  $C^*$ -algebra having no tracial state, then  $A$  has a tracial state (but is not finite), and the reduced crossed product of the trivial action of a finite group on  $A$  is nuclear and has property  $T$ . We will see at the end of this article that one cannot weaken the amenability assumption of  $G$  neither.

**Theorem 3.** *Let  $G$  be an amenable discrete group and  $A$  be a finite nuclear unital  $C^*$ -algebra. If there is an action  $\alpha$  of  $G$  on  $A$  such that  $A \rtimes_{\alpha,r} G$  has property  $T$ , then  $G$  is finite and  $A$  is finite dimensional.*

**Proof:** Set  $I_\alpha := \bigcap_{\tau \in T_\alpha(A)} \ker \pi_\tau$  and  $A_\alpha := A/I_\alpha$ . Denote  $B := A \rtimes_{\alpha,r} G$ . The finiteness assumption of  $A$  and Lemma 1(c) imply that  $I_\alpha \neq A$  and that  $T(B) \neq \emptyset$  (see also Lemma 1(a)). Hence,  $G$  is finite (by

Proposition 2). Moreover, the argument of Proposition 2 tells us that  $I_\alpha$  is  $\alpha$ -invariant and  $B_\alpha := A_\alpha \rtimes_{\beta,r} G$  is finite dimensional. Therefore, it suffices to show that  $I_\alpha = \{0\}$ .

Suppose on the contrary that  $I_\alpha \neq \{0\}$ . By [2, Theorem 5.1], we know that  $B \cong B_0 \oplus B_1$ , where  $B_0$  is finite dimensional and  $T(B_1) = \emptyset$ . Thus,  $I_\alpha \rtimes_{\alpha,r} G = J_0 \oplus J_1$ , with  $J_k$  being a closed ideal of  $B_k$  for  $k \in \{0, 1\}$ . The short exact sequence

$$0 \rightarrow I_\alpha \rightarrow A \rightarrow A_\alpha \rightarrow 0,$$

induces a short exact sequence concerning their full crossed products, which coincide with the reduced crossed products because  $G$  is amenable. From this, we obtain

$$B_\alpha = B / (I_\alpha \rtimes_{\alpha,r} G) = B_0 / J_0 \oplus B_1 / J_1.$$

Hence,  $B_1 / J_1$  is a quotient  $C^*$ -algebra of the finite dimensional  $C^*$ -algebra  $B_\alpha$ , which implies  $J_1 = B_1$  (otherwise,  $B_1$  will have a tracial state). Consequently,  $B_\alpha \cong B_0 / J_0$ , or equivalently,  $B_0 \cong B_\alpha \oplus J_0$  (as  $B_0$  is finite dimensional). This gives

$$B \cong B_\alpha \oplus J_0 \oplus B_1 = B_\alpha \oplus (I_\alpha \rtimes_{\alpha,r} G).$$

Thus,  $I_\alpha \rtimes_{\alpha,r} G$  is unital and so is  $I_\alpha$  (but its identity may not be the identity of  $A$ ).

Now, by the finiteness assumption of  $A$ , one knows that  $T(I_\alpha) \neq \emptyset$ , and Lemma 1(c) produces an element  $\tau \in T_\alpha(I_\alpha)$ . Let  $\Phi : A \rightarrow I_\alpha$  be the canonical  $G$ -equivariant  $*$ -epimorphism, and define

$$\tau'(a) := \langle \pi_\tau(\Phi(a))\xi_\tau, \xi_\tau \rangle \quad (a \in A).$$

Then  $\tau' \in T_\alpha(A)$  and  $\tau'|_{I_\alpha} = \tau$ . However, the existence of  $\tau'$  contradicts the definition of  $I_\alpha$ .  $\square$

**Corollary 4.** *Let  $G$  be an infinite discrete group and  $\alpha_G$  be the left translation action of  $G$  on  $\ell^\infty(G)$ . The following are equivalent.*

- (1)  $G$  is non-amenable.
- (2)  $\ell^\infty(G) \rtimes_{\alpha_G,r} G$  does not have a tracial state.
- (3)  $\ell^\infty(G) \rtimes_{\alpha_G,r} G$  has strong property  $T$  (see [8]).
- (4)  $\ell^\infty(G) \rtimes_{\alpha_G,r} G$  has property  $T$ .
- (5) There is a finite nuclear unital  $C^*$ -algebra  $A$  and an action  $\alpha$  of  $G$  on  $A$  such that  $A \rtimes_{\alpha,r} G$  has property  $T$ .

**Proof:** If  $G$  is non-amenable, then  $T_{\alpha_G}(\ell^\infty(G)) = \emptyset$  and Lemma 1(a) tells us that Statement (2) holds. On the other hand, if  $\ell^\infty(G) \rtimes_{\alpha_G,r} G$  does not have a tracial state, then [8, Proposition 5.2] gives Statement (3). Moreover, a strong property  $T$   $C^*$ -algebra clearly have property  $T$ . Finally, suppose that  $A \rtimes_{\alpha,r} G$  has property  $T$  but  $G$  is amenable. Then Theorem 3 produces the contradiction that  $G$  is finite.  $\square$

The following comparison of Corollary 4 with the main result of Ozawa in [10] (see also Theorem 5.1.6 and Proposition 5.1.3 of [3]) may be worth mentioning:

a discrete  $G$  is exact if and only if  $\ell^\infty(G) \rtimes_{\alpha_G, r} G$  is nuclear  
(or equivalently, the action  $\alpha_G$  is amenable).

This result tells us that one cannot weaken the amenability assumption of  $G$  in Theorem 3 to an amenable action  $\alpha$  with  $A \rtimes_{\alpha, r} G$  being nuclear, since if  $G$  is an infinite exact non-amenable group, the action of  $G$  on  $\ell^\infty(G)$  is amenable, and the reduced crossed product has property  $T$  and is nuclear.

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